Phase Separation in Asymmetrical Fermion Superfluids

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Motivated by recent developments on cold atom traps and high density QCD we consider fermionic systems composed of two particle species with different densities. We argue that a mixed phase composed of normal and superfluid components is the energetically favored ground state. We suggest how this phase separation can be used as a probe of fermion superfluidity in atomic traps.

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The recent interest in two physical systems has revived the study of asymmetric fermionic systems, that is, systems with unequal number density (or chemical potential) for the different species. In high density strongly interacting systems, as it may be found at the core of "neutron" stars, the different quark flavors have different chemical potentials because of their different masses and charges [1–4]. Atomic traps can also provide an example of similar systems if a bias can be introduced when the trap is filled in order to have a larger number of one of the atom species (or hyperfine state of the same atom) [5]. In the symmetrical situation the low energy properties of these systems are dominated by Cooper pair formation. Since it involves the attraction between two fermions with equal and opposite momenta at the their Fermi surface one could imagine that as the Fermi surfaces move apart with increasing asymmetry, the pairing would become weaker and the gap smaller, until superfluidity disappears. What actually happens, the formation of an inhomogeneous mixed phase state, is however more interesting and results from the competition between states with different particle distribution, both in momentum and real space.

Our discussion is valid in a wider class of models but, for definiteness, let us consider a nonrelativistic dilute gas made out of two particle species A and B with chemical potentials μ_A, μ_B and masses M_A, M_B, respectively. At low densities the details of the potential are not probed and their interaction is well described by the pairing Hamiltonian

$$\mathcal{H} - \sum_{i=A,B} \mu_i \mathcal{N}_i = \int \frac{d^3k}{(2\pi)^3} \sum_{i=A,B} \epsilon_k^i \psi_i^\dagger(k)\psi_i(k) + g \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3} \psi_A^\dagger(p)\psi_B^\dagger(-p)\psi_B(-k)\psi_A(k), \tag{1}$$

where $\psi_A^\dagger$, ψ_A are creation and annihilation operators for the A particles and ϵ_k^A is their dispersion relation, that we take to be $\epsilon_k^A = k^2/2M_A - \mu_A$ (and similarly for B). In the mean field approximation, adequate for the low densities considered here, the Hamiltonian can be approximated by

$$\begin{aligned}\mathcal{H} - \sum_{i=A,B} \mu_i \mathcal{N}_i &= -\frac{|\Delta|^2}{g} + \int \frac{d^3k}{(2\pi)^3} \sum_{i=A,B} \epsilon_k^i \psi_i^\dagger(k)\psi_i(k) - \Delta^* \psi_B(-k)\psi_A(k) - \Delta \psi_A^\dagger(k)\psi_B^\dagger(-k) \\ &= -\frac{\Delta^2}{g} + \int \frac{d^3k}{(2\pi)^3}(\epsilon_k^B - E_k^\beta) + \int \frac{d^3k}{(2\pi)^3}[E_k^\alpha \psi_\alpha^\dagger(k)\psi_\alpha(k) + E_k^\beta \psi_\beta^\dagger(k)\psi_\beta(k)],\end{aligned} \tag{2}$$

where $\Delta = -g\int d^3k/(2\pi)^3 \langle \psi_B(-k)\psi_A(k)\rangle = \Delta^*$, $E_k^{\alpha,\beta} = \pm\epsilon_k^- + \sqrt{\epsilon_k^{+2} + \Delta^2}$, $\epsilon^\pm = (\epsilon_k^A \pm \epsilon_k^B)/2$, and the fields ψ_α, ψ_β are defined by

$$\begin{pmatrix} \psi_\alpha(k) \\ \psi_\beta^\dagger(-k) \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} \psi_A(k) \\ \psi_B^\dagger(-k) \end{pmatrix}, \tag{3}$$

with

$$\begin{matrix} u_k^2 \\ v_k^2 \end{matrix} = \frac{1}{2}\left(1 \pm \frac{\epsilon_k^+}{\sqrt{\epsilon_k^{+2} + \Delta^2}}\right). \tag{4}$$

It is straightforward to minimize the diagonalized Hamiltonian shown in Eq. (2). One simply fills the modes with negative $E_k^{\alpha,\beta}$ and leaves the remaining modes empty.

In terms of the original particles A and B and the vacuum state $|0\rangle$, the state above corresponds to having a BCS-like state $[u_k + v_k \psi_A^\dagger(k)\psi_B^\dagger(-k)]|0\rangle$ in the modes k where $E_k^{\alpha,\beta} > 0$, but a state filled with particle B (A) in the modes where $E_k^\beta < 0$ ($E_k^\alpha < 0$). The thermodynamic potential of this state is

$$\begin{aligned}\Omega &= \langle \Psi | \mathcal{H} - \sum_{i=A,B} \mu_i \mathcal{N}_i | \Psi \rangle \\ &= -\frac{M\Delta^2}{2\pi a} + \int \frac{d^3k}{(2\pi)^3}[\theta(-E_k^\alpha)E_k^\alpha + \theta(-E_k^\beta)E_k^\beta + \epsilon_k^B - E_k^\beta],\end{aligned} \tag{5}$$

where a is the scattering length between particle A and B. It is related to the coupling constant (in dimensional regularization) by $1/g = M/2\pi a$ with $M = M_A M_B/(M_A + M_B)$, the reduced mass.

Let us now consider the case $M_B > M_A$, $p_B > p_A$, where p_i is the Fermi momentum defined by $p_i = \sqrt{2M_i\mu_i}$. For some values of Δ, E_k^β may be negative for momenta $k_1 \le k \le k_2$ where

$$k_{1,2}^2 = \frac{p_A^2 + p_B^2}{2} \pm \frac{1}{2}\sqrt{(p_B^2 - p_A^2)^2 - 16M_A M_B\Delta^2}, \quad (6)$$

while E_k^α is always positive.

We now discuss separately the cases where either the chemical potentials or the densities of each species are kept fixed.

Fixed chemical potentials.—In Fig. 1 we show the thermodynamic potential as a function of Δ for different values of p_A and p_B, keeping the combination $p_0^2/M = p_A^2/M_A + p_B^2/M_B$ fixed, computed from a numerical evaluation of Eq. (5). An analytical expression valid for $\Delta \ll \mu_{A,B}$ is available but it is not very enlightening.

For large enough Δ, $k_{1,2}^2$ are not real, $E_k^{\alpha,\beta}$ is always positive, and the thermodynamic potential is unchanged from the $p_B = p_A$ case. However, for $\Delta < (p_B^2 - p_A^2)/(4\sqrt{M_A M_B})$, $k_{1,2}^2$ are positive and the thermodynamic potential can be lowered by filling the states between k_1 and k_2 with β-type quasiparticles. The $\Delta = 0$ state, in particular, has its thermodynamic potential lowered with increasing $p_B^2 - p_A^2$ and at some point becomes smaller than the previous minimum with $\Delta = \Delta_0$ corresponding to the BCS phase. The result is that, for fixed p_0 and increasing $p_B^2 - p_A^2$, there is a first-order phase transition between the superfluid and the normal state, as it has been noticed in different physics contexts (see, for instance Refs. [6–9]). These results can be understood in very simple physical terms. Suppose we start in the BCS phase with $\mu_A = \mu_B$ and increase p_B. By absorbing one B particle and eliminating one A the system on one hand reduces its thermodynamic potential Ω due to the $-\mu_A n_A - \mu_B n_B$ term but on the other hand it increases Ω by destroying two pairs. This is energetically favorable if, and only if, the difference in chemical potentials is large enough (or the gap small enough). Until that point the BCS state with equal number of particles remains the ground state, unchanged despite the variations in chemical potential.

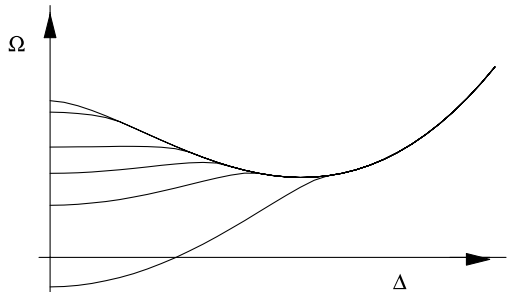

FIG. 1. Thermodynamic potential for different values of p_B and p_A (constant p_0). The top curve corresponds to $p_A = p_B$ and the lower curves correspond to increasing values of $|p_B^2 - p_A^2|$.

In addition to the stable (or metastable) normal and BCS phases there is, for some values of the chemical potentials μ_A and μ_B, an unstable phase (referred to here as "Sarma state") corresponding to a maximum of Ω as a function of Δ situated between the BCS minimum Δ_0 and the normal phase at $\Delta = 0$ (first pointed out in Ref. [10]). The combination of parameters necessary for the existence of this phase can be found considering the gap equation:

$$0 = \frac{d\Omega}{d\Delta^2} = -\frac{M}{2\pi a} - \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_k^{+2} + \Delta_s^2}} + \frac{1}{2}\int_{k_1}^{k_2} \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_k^{+2} + \Delta_s^2}} = -\frac{M}{2\pi a} - \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_k^{+2} + \Delta_0^2}}, \quad (7)$$

where the BCS gap is given by $2(p_0^2/M) \times \exp[-\pi/(2p_0|a|) - 2]$ in the mean field approximation. For small values of the gaps $\Delta_0, \Delta_s \ll \mu_A, \mu_B$ the integrals can be approximated and it is found that [10–12]

$$\Delta_s \cong \sqrt{\Delta_0\left(\frac{p_B^2 - p_A^2}{2\sqrt{M_A M_B}} - \Delta_0\right)}. \quad (8)$$

The Sarma state gap is in the range $0 \le \Delta_s \le \Delta_0$, where the upper bound is set by the condition for the existence of real values of $k_{1,2}^2$. We then have

$$\Delta_0 \le \frac{p_B^2 - p_A^2}{2\sqrt{M_A M_B}} \le 2\Delta_0. \quad (9)$$

The condition in Eq. (9), for fixed p_A and p_B, can thus be seen as determining a window for the values of a supporting the Sarma state [13]. Also notice that when $p_B^2 - p_A^2$ reaches its *largest* value allowing for a Sarma state ($p_B^2 - p_A^2 = 4\sqrt{M_A M_B}\Delta_0$), k_1 equals k_2 and the difference in particle densities $n_B - n_A$ approaches zero, corresponding to a BCS state. In the other limit, $p_B^2 - p_A^2 = 2\sqrt{M_A M_B}\Delta_0$, the gap Δ_s vanishes and the Sarma state reduces to the normal phase.

Fixed particle number.—The previous discussion regarding the stability of different phases was made under the assumption that the particle numbers n_A and n_B are allowed to change. We discuss now the situation where they are fixed. In the BCS phase the particle number densities n_A and n_B are the same, as can be readily seen by taking derivatives of Eq. (5) in relation to μ_A and μ_B, while the particle numbers in the normal phase can be

different. The Sarma phase can also accommodate $n_A \neq n_B$ so the question arises: What is the ground state of the system when the particle densities are fixed and different from each other? This question has been revived in Refs. [13,14] where it was argued that the Sarma phase (named there "interior gap"' state) could be the ground state in the case of fixed particle numbers n_A, n_B. This state would have fascinating properties, being at the same time a Fermi liquid (with two Fermi surfaces corresponding to k_1 and k_2) *and* a superfluid.

The question can be answered by finding the state with the smallest energy (not the thermodynamic potential Ω). We compare here the normal, Sarma, and a mixed inhomogeneous phase composed of bubbles of normal phase in a sea of the BCS phase. The energy for the normal and the BCS phase is, for small values of the gap, given by

$$E_N(n_A, n_B) = \frac{(6\pi^2 n_A)^{5/3}}{20\pi^2 M_A} + \frac{(6\pi^2 n_B)^{5/3}}{20\pi^2 M_B}, \qquad E_{\rm BCS}(n_A = n_B = n) = \frac{(6\pi^2 n)^{5/3}}{20\pi^2 M} - \frac{M\Delta_0^2(n)}{2\pi^2}(6\pi^2 n)^{1/3}. \tag{10}$$

A similar expression can be derived for the energy of the Sarma phase but its form is not very enlightening and it will not be needed below.

The mixed phase is an inhomogeneous phase where a fraction x of the space is in the normal phase with A and B particle densities equal to $\bar{n}_A$ and $\bar{n}_B$, while the remaining $1 - x$ fraction is in the BCS phase with a common density for both species equal to $\bar{n}$. The densities in each component are adjusted in such a way that the overall average densities have given prescribed values n_A, n_B; that is, $n_A = x\bar{n}_A + (1 - x)\bar{n}$ and similarly for the particles B. The most favored mixed state for given n_A, n_B is the one with the smallest energy:

$$E_{\rm mix}(n_A, n_B) = \min_{x,\bar{n}}\left[(1 - x)E_{\rm BCS}(\bar{n}) + xE_N\left(\frac{n_A - (1 - x)\bar{n}}{x}, \frac{n_B - (1 - x)\bar{n}}{x}\right)\right]. \tag{11}$$

We have disregarded the interface energy between the two components, as those are small for large enough systems.

There are two limiting cases where the comparison between the mixed and Sarma phases can be done analytically, corresponding to parameters where the inequalities in Eq. (9) are saturated. If $p_B^2 - p_A^2 = 2\sqrt{M_A M_B}\Delta_0$ (Δ_s=0), the Sarma state reduces to the normal state. In this case its energy is given by

$$E_S = E_N = \frac{(6\pi^2)^{5/3}}{20\pi^2}\left(\frac{n_A^{5/3}}{M_A} + \frac{(n_A + \delta n)^{5/3}}{M_B}\right) \cong \frac{(6\pi^2 n_A)^{5/3}}{20\pi^2 M}\left[1 + \frac{5M}{3M_B}\frac{\delta n}{n_A} + \frac{5M}{9M_B}\left(\frac{\delta n}{n_A}\right)^2 + \mathcal{O}(\delta n^3/n_A^3)\right], \tag{12}$$

where $\delta n = n_B - n_A$ is assumed to be small, $\delta n \ll n_A$.

An upper bound on $E_{\rm mix} - E_S$ can be obtained by setting the density of the BCS component of the mixed phase $\bar{n} = n_A$ and minimizing in relation of x. We have

$$E_{\rm mix} - E_S \cong -(1 - x)(6\pi^2 n_A)^{1/3}\frac{M\Delta_0^2(p_A)}{2\pi^2} + \frac{(6\pi^2 n_A)^{5/3}}{36\pi^2 M_B}\frac{\delta n^2}{n_A^2}\left(\frac{1}{x} - 1\right) + \mathcal{O}(n_A^{-4/3}\delta n^3), \tag{13}$$

whose minimization yields

$$x = x_{\rm min} = \sqrt{\frac{(6\pi^2 n_A)^{4/3}}{18MM_B\Delta_0^2(p_A)}\frac{\delta n^2}{n_A^2}}, \tag{14}$$

$$E_{\rm mix} - E_S \cong -(6\pi^2 n_A)^{1/3}\frac{M\Delta_0^2(p_A)}{2\pi^2}(1 - x_{\rm min})^2 < 0.$$

Numerical calculations show that the upper bound above is close to the actual minimum. Notice that $x_{\rm min} \cong \sqrt{(M_A + M_B)/2M_B} < 1$ as it should be. Equation (13) shows that the mixed phase is energetically favored compared to the Sarma phase in one extreme of the window in Eq. (9).

The other simple limit to analyze corresponds to $p_B^2 - p_A^2 = 4\sqrt{M_A M_B}\Delta_0$, in which case $\Delta_s = \Delta_0$, $k_1 = k_2$, $n_A = n_B$, and the Sarma phase reduces to the BCS phase. In this case the mixed phase reduces to the BCS phase too and the energies of both the Sarma and mixed phase are equal to each other. For intermediate values of $p_B^2 - p_A^2$ [still satisfying the constraint in Eq. (9)], the difference $E_{\rm mix} - E_S$ interpolates between these two extremes, as Fig. 2 exemplifies. We find that for all reasonable values of the parameters (that is, where the mean field analysis should apply), and for fixed particle numbers n_A and n_B, the mixed phase has a smaller energy than the Sarma phase.

Fixed total density.—Another interesting situation arises when the total number of particles is fixed, but conversions between particles of types A and B are allowed. This is relevant for the physics of high density quark matter where weak interactions can change the flavor of the quarks. In this situation $n = n_A + n_B$ and $\delta\mu = (\mu_B - \mu_A)/2$ are fixed and the thermodynamic function that should be minimized is $E - \delta\mu(n_B - n_A)$. The nonrelativistic formulation presented here is more appropriate for cold atoms. In cold atom traps, only species with nearly equal masses can convert into one another. Further, we consider $\delta\mu = 0$ which is relevent for cold atoms for convenience. It is straightforward to see, then, that the condition for the existence of Sarma state Eq. (9) is not

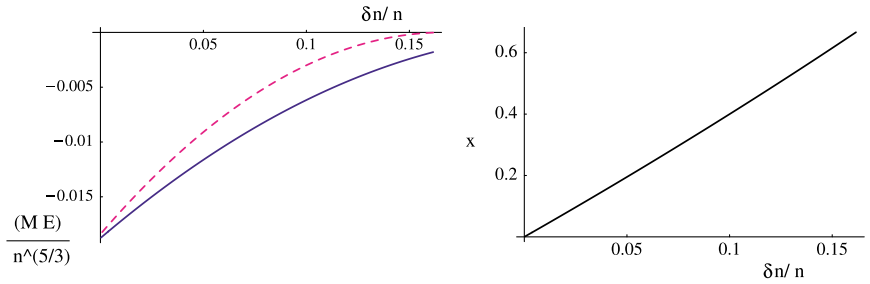

FIG. 2 (color online). Left: Energy of the Sarma (dashed curve) and mixed phase (solid curve) relative to the normal phase as a function of $\delta n/n$ where $n = (n_A + n_B)/2$ with $M_B = 7M_A$. The values of n_A and n_B were obtained from the Sarma phase for a fixed a, and $|a|p_A = 0.59$, $0.63 \leq |a|p_B \leq 0.65$. Right: The fraction of normal state x as a function of $\delta n/n$ with the same masses and scattering length.

satisfied. The same also holds when $\delta\mu = 0$ and $M_B \gg M_A$. The composition in the mixed phase is easy to calculate when $\delta\mu = 0$. The most favored composition minimizes

$$E(N = n_A + n_B) = \mathrm{Min}_{x,\bar{n}}\left[(1-x)E_{\mathrm{BCS}}(\bar{n}) + \frac{(6\pi 2)^{5/3}}{20\pi 2x^{2/3}}\frac{[n-2(1-x)\bar{n}]^{5/3}}{(M_A^{3/2}+M_B^{3/2})^{2/3}}\right]. \quad (15)$$

When $M_A = M_B$, BCS (with $x = 0$) is the favored state and when $M_B \gg M_A$ normal phase ($x = 1$) with only particles of species B is favored.

We have considered Fermi gases made up of two species, when an asymmetry on their densities or chemical potentials tries to push their Fermi surfaces apart, making pairing more difficult. We find that with either both chemical potentials or both densities fixed, the most likely ground state is a mixed phase composed of bubbles of an asymmetric normal state immersed in a sea of the symmetric BCS phase. It is worth mentioning that the Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) state [15] (in which the condensate varies in space) has lower free energy than the normal and the BCS states. However, the LOFF state can exist only in a very narrow window of asymmetry for the chemical potentials and we ignored this posibility in our discussion.

The space segregation of the excess particles in the mixed phase suggests a possible way of detecting superfluidity in atomic traps, especially where large gaps are expected as in the case of "resonance superfluidity" [16]. If an optical trap can be filled with an excess number of one of the hyperfine states, the resulting ground state can be imaged in a way that discriminates between them and the bubble structure may become visible. A high concentration of the denser particle species will accumulate at some point(s) in the trap. The division of the space between the BCS and the normal components is determined by the value of the gap thus, by studying its variation with the variation of the asymmetry, we can infer the existence of superfluidity and even the value of the gap, which is currently an outstanding problem. A better understanding of the surface tension of the interface between the two components is necessary to make this proposal fully quantitative. The qualitative arguments discussed above should be valid even if weak coupling BCS theory is not, as is the case in the experiments with ^{6}Li [16].

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